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Solutions Manual to Walter Rudin's *Principles of Mathematical Analysis*

Roger Cooke, University of Vermont

Chapter 2

Basic Topology

Exercise 2.1 Prove that the empty set is a subset of every set.

Solution. Let $\varnothing$ denote the empty set, and let E be any set. The statement $\varnothing \subset E$ is equivalent to the statement, "If $x \in \varnothing$, then $x \in E$." Since the hypothesis of this if-then statement is false, the implication is true, and we are done.

Exercise 2.2 A complex number z is said to be *algebraic* if there are integers $a_0, \ldots, a_n$, not all zero, such that

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0.$$

Prove that the set of all algebraic numbers is countable. *Hint:* For every positive integer N there are only finitely many equations with

$$n + |a_0| + |a_1| + \cdots + |a_n| = N.$$

Solution. Following the hint, we let A_N be the set of numbers satisfying one of the equations just listed with $n+|a_0|+|a_1|+\cdots+|a_n| = N$. The set A_N is finite, since each equation has only a finite set of solutions and there are only finitely many equations satisfying this condition. By the corollary to Theorem 2.12 the set of algebraic numbers, which is the union $\bigcup_{N=2}^{\infty} A_N$, is at most countable. Since all rational numbers are algebraic, it follows that the set of algebraic numbers is exactly countable.

Exercise 2.3 Prove that there exist real numbers which are not algebraic.

Solution. By the previous exercise, the set of real algebraic numbers is countable. If every real number were algebraic, the entire set of real numbers would be countable, contradicting the remark after Theorem 2.14.

Exercise 2.4 Is the set of irrational real numbers countable?

Answer. No. If it were, the set of all real numbers, being the union of the rational and irrational numbers, would be countable.

Exercise 2.5 Construct a bounded set of real numbers with exactly three limit points.

Solution. Let E be the set of numbers of the form $a + \frac{1}{n}$, where $a \in \{1, 2, 3\}$ and $n \in \{2, 3, 4, 5, \ldots, \}$. It is clear that $\{1, 2, 3\} \subseteq E'$, since every deleted neighborhood of 1, 2, or 3, contains a point in E. Conversely, if $x \notin \{1, 2, 3\}$, let $\delta = \min\{|x-1|, |x-2|, |x-3|\}$. Then the set U of y such that $|x - y| < \delta/2$ contains at most a finite number of points of E, since the set $V = (1, 1 + \frac{\delta}{2}) \cup (2, 2 + \frac{\delta}{2}) \cup (3, 3 + \frac{\delta}{2})$ is disjoint from U, and V contains all the points of the set E except possibly the finite set of points $a + \frac{1}{n}$ for which $n \leq \frac{2}{\delta}$. If $p_1, \ldots, p_r$ are the points of E in U, let η be the minimum of $\frac{\delta}{2}$ and the $|x - p_j|$ for which $x \neq p_j$. Then the set W of points y such that $|y - x| < \eta$ contains no points of E except possibly x. Hence $x \notin E'$. Thus $E' = \{1, 2, 3\}$.

Exercise 2.6 Let E' be the set of all limit points of a set E. Prove that E' is closed. Prove that E and $\overline{E}$ have the same limit points. (Recall that $\overline{E} = E \cup E'$.) Do E and E' always have the same limit points?

Solution. To show that E' is closed, we shall show that $(E')' \subseteq E'$. In fact, we shall show the even stronger statement that $\left(\overline{E}\right)' \subseteq E'$. To do this let $x \in \left(\overline{E}\right)'$, and let $r > 0$. We need to show that $x \in E'$; that is, since $r > 0$ is arbitrary, we need to find a point $z \in E$ with $0 < d(z, x) < r$. There certainly is a point y of $\overline{E}$ such that $0 < d(y, x) < r$. If $y \in E$, we can take $z = y$, and we are done. If $y \notin E$, then $y \in E'$. Let $s = \min\big(d(x, y), r - d(x, y)\big)$, so that $s > 0$. Since $y \in E'$, there exists $z \in E$ with $0 < d(x, z) < s$. But it then follows that $d(z, x) \geq d(x, y) - d(x, z) > 0$ and $d(z, x) \leq d(x, y) + d(y, z) < d(x, y) + r - d(x, y) = r$, and we are done in any case.

To show that E and $\overline{E}$ have the same limit points, we need only show the converse of the preceding containment. But this is easy. Suppose $x \in E'$. Since every deleted neighborhood of x contains a point of E, *a fortiori* every deleted neighborhood of x contains a point of $\overline{E}$. Hence $E' \subseteq (\overline{E})'$.

Certainly E and E' may have different sets of limit points. For example if $E = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\}$, then $E' = \{0\}$, while $(E')' = \varnothing$.

Exercise 2.7 Let $A_1, A_2, A_3, \ldots$ be subsets of a metric space.

(*a*) If $B_n = \cup_{i=1}^{n} A_i$, prove that $\overline{B}_n = \cup_{i=1}^{n} \overline{A}_i$, for $n = 1, 2, 3, \ldots$.

(*b*) If $B = \cup_{i=1}^{\infty} A_i$, prove that $\overline{B} \supset \cup_{i=1}^{\infty} \overline{A}_i$.

Show, by an example, that this inclusion can be proper.

Solution. We first show that $\overline{E \cup F} = \overline{E} \cup \overline{F}$, which follows from the stronger fact that $(E \cup F)' = E' \cup F'$. To show this, in turn, we note that if $x \in E'$, then certainly $x \in (E \cup F)'$, and similarly if $x \in F'$. Hence $E' \cup F' \subseteq (E \cup F)'$. To show the converse, suppose $x \notin E' \cup F'$. Then there is a positive number r such that there is no element y of E with $0 < d(x,y) < r$, and a positive number s such that there is no element y of F with $0 < d(x,y) < s$. Hence if $t = \min(r,s)$, then $t > 0$, and there is no element y of $E \cup F$ with $0 < d(x,y) < t$. Therefore $x \notin (E \cup F)'$.

The general result of (a) now follows easily by induction on n, since

$$\begin{aligned}
\overline{B}_n &= \overline{\cup_{i=1}^n A_i} \\
&= \overline{A_1 \cup \cup_{i=2}^n A_i} \\
&= \overline{A}_1 \cup \overline{\cup_{i=2}^n A_i} \\
&= \overline{A}_1 \cup \cup_{i=2}^n \overline{A}_i \\
&= \cup_{i=1}^n \overline{A}_i.
\end{aligned}$$

Part (b) amounts to the trivial observation that, since $B \supseteq A_i$ for all i, then $\overline{B} \supseteq \overline{A}_i$ for all i, and so

$$\overline{B} \supseteq \cup_{i=1}^\infty \overline{A}_i.$$

If we let $A_i = \{r_i\}$, where $\{r_1, r_2, \ldots, r_n, \ldots\}$ is an enumeration of the rational numbers, then B is the full set of rational numbers. Hence $\overline{B} = R^1$, while $\overline{A}_i = A_i$ for each i, i.e., $\cup \overline{A}_i$ is the set of rational numbers.

Exercise 2.8 Is every point of every open set $E \subset R^2$ a limit point of E. Answer the same question for closed sets in R^2.

Answer. Yes. Every point of an open set E is a limit point of E. To see this, let E be an open set in R^2, let $(x_1, x_2) \in E$, let s be such that $(y_1, y_2) \in E$ if $\sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2} < s$, and let $r > 0$. Then the point $(z_1, z_2) = (x_1 + \frac{1}{2}\min(r,s), x_2)$ belongs to E and satisfies $0 < \sqrt{(z_1 - x_1)^2 + (z_2 - x_2)^2} < r$.

There are closed sets for which this statement is not true. For example, any finite set E is closed, yet $E' = \varnothing$ for a finite set.

Exercise 2.9 Let E° denote the set of all interior points of a set E.

(a) Prove that E° is always open.

(b) Prove that E is open if and only if $E^\circ = E$.

(c) If $G \subset E$ and G is open, prove that $G \subset E^\circ$.

(d) Prove that the complement of E° is the closure of the complement of E.

(e) Do E and $\overline{E}$ always have the same interiors?

(*f*) Do E and E° always have the same closures?

Solution. (*a*) Let $x \in E^\circ$. Then there exists $r > 0$ such that $y \in E$ if $d(x,y) < r$. We claim that in fact $y \in E^\circ$ if $d(x,y) < r$, so that $x \in (E^\circ)^\circ$. Indeed if $d(x,y) < r$, let $s = r - d(x,y)$, so that $s > 0$. Then if $d(z,y) < s$, we have (by the triangle inequality) $d(x,z) < r$, and so $z \in E$. By definition this means $y \in E^\circ$. Since y was any point with $d(x,y) < r$, it follows that all such points are in E°, and so $x \in (E^\circ)^\circ$.

(*b*) By definition E is open if and only if each of its points is an interior point, which says precisely that $E = E^\circ$.

(*c*) If $G \subset E$ and G is open, then $G = G^\circ \subseteq E^\circ$.

(*d*) Part (*c*) shows that E° is the largest open set contained in E, i.e., the union of all open sets contained in E. Hence its complement is the intersection of all closed sets containing the complement of E, and this, by Theorem 2.27 (*c*), is the closure of the complement of E.

(*e*) Emphatically not. If E is the rational numbers in the space R^1, then $E^\circ = \varnothing$, while $\overline{E} = R^1$, so that the interior of $\overline{E}$ is R^1.

(*f*) Emphatically not. If E is the rational numbers in the space R^1, then $\overline{E} = R^1$, while $E^\circ = \varnothing$, so that $\overline{E^\circ} = \varnothing$.

Exercise 2.10 Let X be an infinite set. For $p \in X$ and $q \in X$, define

$$d(p,q) = \begin{cases} 1, & (\text{if } p \neq q), \\ 0, & (\text{if } p = q). \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

Solution. It is obvious that $d(p,q) > 0$ if $p \neq q$ and $d(p,p) = 0$; likewise it is obvious that $d(p,q) = d(q,p)$. To show the triangle inequality $d(x,z) \leq d(x,y) + d(y,z)$, note that the maximal value of the left-hand side is 1, and can be attained only if $x \neq z$. In that case y cannot be equal to both x and z, so that at least one term on the right-hand side is also 1.

Each one-point set is open in this metric, since $B_{\frac{1}{2}}(x) \subseteq \{x\}$. Therefore every set, being the union of all its one-point subsets, is open. Hence every set, being the complement of its complement, is also closed. Only finite sets are compact, since any infinite subset has an open covering (by the union of its one-point subsets) that cannot be reduced to a finite subcovering.

Exercise 2.11 For $x \in R^1$ and $y \in R^1$, define

$$\begin{aligned} d_1(x,y) &= (x-y)^2, \\ d_2(x,y) &= \sqrt{|x-y|}, \end{aligned}$$

$$\begin{aligned} d_3(x,y) &= |x^2-y^2|, \\ d_4(x,y) &= |x-2y|, \\ d_5(x,y) &= \frac{|x-y|}{1+|x-y|}, \end{aligned}$$

Determine, for each of these, whether it is a metric or not.

Solution. The function $d_1(x,y)$ fails the triangle inequality condition, since

$$d_1(0,1)+d_1(1,2) = 1+1 = 2 < 4 = d_1(0,2).$$

The function $d_2(x,y)$ meets the triangle inequality condition, since

$$\sqrt{|x-z|} \leq \sqrt{|x-y|} + \sqrt{|y-z|},$$

as one can easily see by squaring both sides. Hence d_2 is a metric.

The function $d_3(x,y)$ fails the positivity condition, since $d_2(1,-1)=0$. (Restricted to $[0,\infty)$, d_3 would be a metric.)

Since $d_4(1,\frac{1}{2})=0$, the function $d_4(x,y)$ likewise fails the positivity condition. It also fails the symmetry condition, since $d_4(x,y) \neq d_4(y,x)$ in general.

The function $d_5(x,y)$ is a metric. In fact we can prove more generally that if $d(x,y)$ is a metric, so is $\rho(x,y) = \frac{d(x,y)}{1+d(x,y)}$. It is obvious that ρ meets the nonnegativity and symmetry requirements, and we need only verify the triangle inequality, which in this case says that

$$\frac{d(x,z)}{1+d(x,z)} \leq \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)}.$$

To do this, let $a=d(x,z)$, $b=d(x,y)$, and $c=d(y,z)$. We need to show that if $a \leq b+c$, then

$$\frac{a}{1+a} \leq \frac{b}{1+b} + \frac{c}{1+c}.$$

Clearing out the denominators, we find this inequality to be equivalent to

$$a+ab+ac+abc \leq b+c+ab+ac+2bc+2abc,$$

which is clearly true.

Exercise 2.12 Let $K \subset R^1$ consist of 0 and the numbers $1/n$, for $n=1$, 2, 3,... . Prove that E is compact directly from the definition without using the Heine-Borel theorem.

Solution. Suppose $K \subset U_\alpha$, where U_α is open. Then 0 must be in some set U_{α_0}. Since U_{α_0} is open, there exists $\delta > 0$ such that $(-\delta,\delta) \subset U_{\alpha_0}$. In particular $1/n \in U_{\alpha_0}$ if $n > \frac{1}{\delta}$. Let N be the largest integer in $\frac{1}{\delta}$, and let α_j, $j=1,\ldots,N$, be such that $\frac{1}{j} \in U_{\alpha_j}$. Then $K \subset \bigcup_{j=0}^{N} U_{\alpha_j}$.

Exercise 2.13 Construct a compact set of real numbers whose limit points form a countable set.

Solution. Let $K = \{0\} \cup \{\frac{1}{n} : n = 1, 2, \ldots\} \cup \{\frac{1}{m} + \frac{1}{n} : n = m, m+1, \ldots; m = 1, 2, \ldots\}$. It is clear that 0 and the points $\frac{1}{m}$ are limit points of K. We need only show that these are all the limit points. Since $x \geq 0$ for all $x \in K$ and for any positive number ε there is only a finite set of numbers in K larger than $1 + \varepsilon$, it is clear that no negative number and no number larger than 1 can be a limit point of K. Hence we need only consider positive numbers x satisfying $0 < x < 1$. If x is such a number and x is not one of the points $\frac{1}{m}$, let p be such that $\frac{1}{p+1} < x < \frac{1}{p}$, and let $\varepsilon = \frac{1}{2}\min(x - \frac{1}{p+1}, \frac{1}{p} - x)$. The intersection of the set K with the interval $(x - \varepsilon, x + \varepsilon)$ is contained in the set of points $\{\frac{1}{p+1} + \frac{1}{k} : p+1 \leq k < \frac{1}{\varepsilon}\} \cup \{\frac{1}{m} + \frac{1}{n} : m \leq n < \frac{1}{p+1} - \frac{1}{p+2}; m = p+2, \ldots, 2p+2\}$, which is a finite set. Therefore x cannot be a limit point of K.

Exercise 2.14 Give an example of an open cover of the segment $(0, 1)$ which has no finite subcover.

Solution. Let $A_n = (\frac{1}{n}, \frac{n-1}{n})$, $n = 3, 4, \ldots$. If $0 < x < 1$, then $x \in A_n$ if $n > 1/\min(x, 1-x)$, so that $\bigcup\limits_{n=3}^{\infty} A_n$ covers $(0, 1)$. However, the union any finite collection $\{A_1, \ldots, A_N\}$ is an interval $(\frac{1}{k}, \frac{k-1}{k})$, which fails to contain the point $\frac{1}{2k}$.

Exercise 2.15 Show that Theorem 2.36 and its Corollary become false (in R^1, for example) if the word "compact" is replaced by "closed" or "bounded."

Solution. Theorem 2.36 asserts that if a family of closed subsets has the finite intersection property (any finite collection of the sets has a non-empty intersection), then the entire family has a non-empty intersection. To see why this fails for sets that are merely bounded or merely closed, let $A_n = (0, \frac{1}{n})$ and $B_n = [n, \infty)$. The sets A_n are bounded, and the sets B_n are closed. Any finite intersection of the A_n's is nonempty, and any finite intersection of the B_n's is nonempty, yet $\bigcap\limits_{n=1}^{\infty} A_n = \varnothing = \bigcap\limits_{n=1}^{\infty} B_n$.

The corollary asserts that a nested sequence of nonempty compact sets has a nonempty intersection, and the examples just given show that compactness cannot be replaced by either closedness or boundedness.

Exercise 2.16 Regard Q, the set of all rational numbers, as a metric space, with $d(p, q) = |p - q|$. Let E be the set of all $p \in Q$ such that $2 < p^2 < 3$. Show that E is closed and bounded in Q, but that E is not compact. Is E open in Q?

Solution. Suppose $x \in Q \setminus E$. We claim that x is an interior point of the complement of E (which by definition means E is closed). In fact if $x^2 \le 2$, then $x^2 < 2$, since there is no rational number whose square is 2. If $x = 0$, let $\delta = 1$; otherwise let $\delta = \min(\sqrt{\frac{2-x^2}{3}}, \frac{2-x^2}{3|x|})$. Then if $y \in (x-\delta, x+\delta)$, we have $y^2 < 2$. This is obvious if $x = 0$ and $\delta = 1$. In the other case let $y = x+h$, where $|h| < \delta$. Then $y^2 = x^2 + 2xh + h^2 < x^2 + 2|x|\delta + \delta^2 < x^2 + \frac{2}{3}(2-x^2) + \frac{2-x^2}{3} = 2$. Hence x is an interior point of the complement of E.

Similarly suppose $x^2 \ge 3$. Since there is no rational number whose square is 3, we must have $x^2 > 3$. Since $x \ne 0$, we let $\delta = \frac{x^2-3}{2|x|}$. Then if $y \in (x-\delta, x+\delta)$, we have $y^2 > 3$. For since $y = x+h$, with $|h| < \delta$, and so $y^2 = x^2 + 2xh + h^2 > x^2 - 2|x|\delta = 3$. Thus again x is an interior point of the complement of E.

Hence in all cases $Q \setminus E$ is open, so that E is closed.

That E is bounded is obvious, since $E \subset [-2, 2]$.

To show that E is not compact, let $U_n = \{p : 2 < p^2 < 3 - \frac{1}{n}\}$, $n = 2, 3, \ldots$. The argument that will be used below to show that E is open shows that U_n is open. The sets U_n cover E, but no finite collection of them covers E. Thus E is not compact.

The set E is also open, since if $2 < x^2 < 3$, we can let δ be the minimum of $\sqrt{\frac{3-x^2}{3}}$, $\frac{3-x^2}{3|x|}$, and $\frac{x^2-2}{2|x|}$. Then if $y \in (x-\delta, x+\delta)$, we must have $2 < y^2 < 3$, by the same set of inequalities that was used above.

Exercise 2.17 Let E be the set of all $x \in [0, 1]$ whose decimal expansion contains only the digits 4 and 7. Is E countable? Is E dense in $[0, 1]$? Is E compact? Is E perfect?

Solution. The set E is not countable, since for any hypothetical list of its elements $a_1, a_2, \ldots, a_n, \ldots$ we can always produce an element a of E not in the list by taking the nth digit of a to be 4 if the nth digit of a_n is 7 and equal to 7 if the nth digit of a_n is 4.

The set E is not dense in $[0, 1]$, since $E \subset [0.4, 0.8]$

The set E is closed and bounded, and therefore compact. To show that E is closed, let $x \in [0, 1] \setminus E$, i.e., the decimal expansion of x contains a digit different from 4 and 7. Let the first such digit occur in the nth place (x_n). Let y be any element of E, and let the first digit in which x and y differ be the mth digit ($m \le n$, $x_m \ne y_m$). Then $|x - y| \ge 10^{-m} - \varepsilon$, $\varepsilon \le \sum\limits_{k=m+1}^{\infty} 10^{-k}|x_k - y_k|$. Since $y_k \in \{4, 7\}$ and $x_k \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, it follows that $|x_k - y_k| \le 7$. Hence $\varepsilon \le \frac{7}{9}10^{-m}$, and it follows that $|x - y| \ge \frac{2}{9 \cdot 10^m} \ge \frac{1}{9 \cdot 10^n}$. Thus x is an interior point of $[0, 1] \setminus E$, and so E is closed.

The set E is perfect. For each $x \in E$ and each $\varepsilon > 0$ we can find a point $y \in E$ with $0 < |x - y| < \varepsilon$ by changing the nth digit of x from 4 to 7 or from 7 to 4 in the nth place for any $n > 1 - \log_{10}\varepsilon$. Hence $x \in E'$, i.e., $E \subseteq E'$. Since we already know E is closed, it follows that $E = E'$.

Exercise 2.18 Is there a non-empty perfect set in R^1 which contains no rational number?

Answer. Yes. Let $\{r_1, r_2, \ldots, r_n, \ldots\}$ be the rational numbers in the interval $[-\pi, \pi]$. Let $E_0 = [-\pi, \pi]$. Now assume that E_k has been chosen for $k < n$ in such a way that E_k is a pairwise disjoint union of at most $2^{k+1} - 1$ closed intervals with irrational endpoints, each of *positive* length at most $(\frac{2}{3})^k \pi$ and that E_k does not contain r_j if $j \le k$. (All of these conditions hold trivially for $k = 0$.) Define a set F_{k+1}, which is obtained from E_k by removing first the middle third of each of the intervals that constitute E_k. The result is a set of at most $2^{k+2} - 2$ pairwise disjoint intervals having irrational endpoints, each interval being of length at most $(\frac{2}{3})^{k+1}\pi$. If $r_{k+1} \notin F_{k+1}$, let $E_{k+1} = F_{k+1}$. If $r_{k+1} \in F_{k+1}$, then r_{k+1} is not the endpoint of the interval $I = [a, b]$ of F_{k+1} that it belongs to. Hence let δ be an irrational positive number less than the minimum of $r_{k+1} - a$ and $b - r_{k+1}$, and let E_{k+1} be obtained from F_{k+1} by removing the interval $(r_{k+1} - \delta, r_{k+1} + \delta)$ (which has irrational endpoints). Then E_{k+1} consists of at most $2^{k+2} - 1$ pairwise disjoint closed intervals, each of positive length at most $(\frac{2}{3})^{k+1}\pi$, and each having irrational endpoints.

The sets E_k form a nested sequence of nonempty compact sets. Hence the intersection $E = \bigcap\limits_{k=0}^{\infty}$ is a nonempty compact set. By construction it contains no rational numbers. To show that it is perfect, we merely observe that if $x \in E$, then for each k there is a unique interval $I_k = [a_k, b_k]$, among the finite set of closed intervals constituting the set E_k such that $x \in I_k$. Let $y_k = a_k$ if $a_k \ne x$, otherwise let $y_k = b_k$. In either case $y_k \in E$ (since in our construction no endpoint of any E_k is ever removed) and $|y_k - x| < 2 \cdot 3^{-k}\pi$. Therefore $x \in E'$.

Exercise 2.19 (*a*) if A and B are disjoint closed sets in some metric space X, prove that they are separated.

(*b*) Prove the same for disjoint open sets.

(*c*) Fix $p \in X$, $\delta > 0$, define A to be the set of all $q \in X$ for which $d(p, q) < \delta$, define B similarly with $>$ in place of $<$. Prove that A and B are separated.

(*d*) Prove that every connected space with at least two points is uncountable. *Hint:* Use (*c*).

Solution. (*a*) We are given that $A \cap B = \varnothing$. Since A and B are closed, this means $A \cap \overline{B} = \varnothing = \overline{A} \cap B$, which says that A and B are separated.

(*b*) Since $X \setminus B$ is a closed set containing A, it follows from Theorem 2.27 (*c*) that $X \setminus B \supseteq \overline{A}$, i.e., that $\overline{A} \cap B = \varnothing$. Similarly $A \cap \overline{B} = \varnothing$.

(*c*) The sets A and B are disjoint open sets, hence by part (*b*) they are separated.

(*d*) Let $x \in X$ and $y \in X$, and let $d(x, y) = d > 0$. Then for every $\delta \in (0, d)$, there must be a point z such that $d(x, z) = \delta$. (If not, the sets A and B defined in part (*c*) would separate X.) Hence there is a subset of X that can be placed in one-to-one correspondence with the interval $[0, d]$, and so X is uncountable.

Exercise 2.20 Are closures and interiors of connected sets always connected? (Look at subsets of R^2.)

Answer. The closure of a connected set is connected. Indeed if E is connected and $E \subseteq F \subseteq \overline{E}$, then F is connected. For, suppose $F = G \cup H$, where G and H are separated, nonempty sets. The set E cannot be contained entirely in G. (If it were, since H is nonempty, H would contain a limit point of E, hence a limit point of G, contrary to hypothesis.) For the same reason E cannot be contained entirely in H. Hence $G_1 = E \cap G$ and $H_1 = E \cap H$ are nonempty separated sets such that $E = G_1 \cup H_1$, and E is not connected.

The interior of a connected set may fail to be connected, as we see by letting E be the union of two closed disks in R^2 that are tangent to each other.

Exercise 2.21 Let A and B be separated subsets of some R^k, suppose $\mathbf{a} \in A$, $\mathbf{b} \in B$, and define

$$\mathbf{p}(t) = (1-t)\mathbf{a} + t\mathbf{b}$$

for $t \in R^1$. Put $A_0 = \mathbf{p}^{-1}(A)$, $B_0 = \mathbf{p}^{-1}(B)$. [Thus $t \in A_0$ if and only if $\mathbf{p}(t) \in A$.]

(*a*) Prove that A_0 and B_0 are separated subsets of R^1.

(*b*) Prove that there exists $t_0 \in (0,1)$ such that $\mathbf{p}(t_0) \notin A \cup B$.

(*c*) Prove that every convex subset of R^k is connected.

Solution. (*a*) The definition shows that A_0 and B_0 are disjoint. We need only show that neither contains a limit point of the other. Let x be a limit point of A_0, and suppose $x \in B_0$. This means that for any $\delta > 0$ there exists $t \in A_0$ with $0 < |x-t| < \delta$, $\mathbf{p}(t) = (1-t)\mathbf{b} + t\mathbf{b} \in A$ and $\mathbf{p}(x) = (1-x)\mathbf{a} + x\mathbf{b} \in B$. Now $d(\mathbf{p}(t), \mathbf{p}(x)) = |\mathbf{p}(t) - \mathbf{p}(x)| = |x-t|\,|\mathbf{a}-\mathbf{b}| \le |x-t|\,(|\mathbf{a}| + |\mathbf{b}|) < M\delta$, where $M = |\mathbf{a}| + |\mathbf{b}|$. Since δ is arbitrary, this means that B contains a limit point of A, contrary to hypothesis. This contradiction shows that B_0 contains no limit points of A_0 Likewise A_0 contains no limit points of B_0, and so A_0 and B_0 are separated.

(*b*) If $\mathbf{p}(t) \in A \cup B$ for all $t \in [0,1]$, then $[0,1] \subseteq A_0 \cup B_0$. Hence $[0,1] = G \cup H$, where $G = [0,1] \cap A_0$ and $H = [0,1] \cap B_0$ are both nonempty ($0 \in G$ and $1 \in H$) and separated. This would mean $[0,1]$ is not connected. Therefore $\mathbf{p}(t_0) \notin A \cup B$ for some $t_0 \in [0,1]$, and necessarily $t_0 \in (0,1)$, since $\mathbf{p}(0) = \mathbf{a} \in A$ and $\mathbf{p}(1)\mathbf{b} \in B$.

(*c*) By definition a convex set C is one for which the mapping $\mathbf{p}$ has the property $\mathbf{p}(t) \in C$ for all $t \in [0,1]$ provided $\mathbf{p}(0) = \mathbf{a} \in C$ and $\mathbf{p}(1) = \mathbf{b} \in C$. Hence by part (*b*) there cannot be separated nonempty sets A and B such that $C = A \cup B$.

Exercise 2.22 A metric space is called *separable* if it contains a countable dense subset. Show that R^k is separable. *Hint:* Consider the set of points which have only rational coordinates.

Solution. We need to show that every non-empty open subset E of R^k contains a point with all coordinates rational. Now E contains a ball $B_r(\mathbf{x})$, and this ball contains all points $\mathbf{y}$ such that $(x_j - y_j)^2 < \frac{1}{k}$ for $j = 1, 2, \ldots, k$. Each interval $(x_j - \frac{1}{k}, x_j + \frac{1}{k})$ contains a rational number r_j, and so the point $\mathbf{r} = (r_1, \ldots, r_k)$ belongs to E. Thus E contains a point with only rational coordinates.

Exercise 2.23 A collection $\{V_\alpha\}$ of open sets of X is said to be a *base* for X if the following is true: For every $x \in X$ and every open set $G \subset X$ such that $x \in G$, we have $x \in V_\alpha \subset G$ for some α. In other words, every open set in X is the union of a subcollection of $\{V_\alpha\}$.

Prove that every separable metric space has a *countable* base. *Hint:* Take all neighborhoods with rational radius and center in some countable dense subset of X.

Solution. Let $\{x_1, x_2, \ldots, x_n, \ldots\}$ be a countable dense subset of X. For each positive integer m and each positive rational number r let $V_{m,r} = \{y : d(y, x_m) < r\}$. The collection $V_{m,r}$ is countable.

Let $x \in X$, and let G be any open subset of X with $x \in G$. Then there exists $\delta > 0$ such that $B_\delta(x) \subset G$. The open ball $B_{\frac{\delta}{2}}(x)$ contains a point x_k for some k. Let r be a rational number such that $d(x_k, x) < r < \frac{\delta}{2}$. Then $x \in B_r(x_k) \subset B_\delta(x) \subset G$, and we are done.

Exerçise 2.24 Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable. *Hint:* Fix $\delta > 0$, and pick $x_1 \in X$. Having chosen $x_1, \ldots, x_j \in X$, choose $x_{j+1} \in X$, if possible, so that $d(x_j, x_{j+1}) \geq \delta$ for $i = 1, \ldots, j$. Show that this process must stop after a finite number of steps, and that X can therefore be covered by finitely many neighborhoods of radius δ. Take $\delta = 1/n$ ($n = 1, 2, 3, \ldots$), and consider the centers of the corresponding neighborhoods.

Solution. Following the hint, we observe that if the process of constructing x_j did not terminate, the result would be an infinite set of points x_j, $j = 1, 2, \ldots$, such that $d(x_i, x_j) \geq \delta$ for $i \neq j$. It would then follow that for any $x \in X$, the open ball $B_{\frac{\delta}{2}}(x)$ contains at most one point of the infinite set, hence that no point could be a limit point of this set, contrary to hypothesis. Hence X is *totally bounded*, i.e., for each $\delta > 0$ there is a *finite* set $x_1, \ldots, x_{N_\delta}$ such that $X = \bigcup_{j=1}^{N_\delta} B_\delta(x_j)$.

Let $x_{n1}, \ldots, x_{nN_n}$ be such that $X = \bigcup_{j=1}^{N_n} B_{\frac{1}{n}}(x_{nj})$, $n = 1, 2, \ldots$. We claim that $\{x_{nj} : 1 \leq j \leq N_n;\, n = 1, 2, \ldots\}$ is a countable dense subset of X. Indeed if $x \in X$ and $\delta > 0$, then $x \in B_{\frac{1}{n}}(x_{nj})$ for some x_{nj} for some $n > \frac{1}{\delta}$, and hence $d(x, x_{nj}) < \delta$. By definition, this means that $\{x_{nj}\}$ is dense in X.

Exercise 2.25 Prove that every compact metric space K has a countable base, and that K is therefore separable. *Hint:* For every positive integer n, there are finitely many neighborhoods of radius $1/n$ whose union covers K.

Solution. It is easier simply to refer to the previous problem. The hint shows that K can be covered by a finite union of neighborhoods of radius $1/n$, and the previous problem shows that this implies that K is separable.

It is not entirely obvious that a metric space with a countable base is separable. To prove this, let $\{V_n\}_{n=1}^{\infty}$ be a countable base, and let $x_n \in V_n$. The points V_n must be dense in X. For if G is any non-empty open set, then G contains V_n for some n, and hence $x_n \in G$. (Thus for a metric space, having a countable base and being separable are equivalent.)

Exercise 2.26 Let X be a metric space in which every infinite subset has a limit point. Prove that X is compact. *Hint:* By Exercises 23 and 24, X has a countable base. It follows that every open cover of X has a *countable* subcover $\{G_n\}_{n=1}$, $n = 1, 2, 3, \ldots$. If no finite subcollection of $\{G_n\}$ covers X, then the complement F_n of $G_1 \cup \cdots \cup G_n$ is nonempty for each n, but $\cap F_n$ is empty. If E is a set which contains a point from each F_n, consider a limit point of E, and obtain a contradiction.

Solution. Following the hint, we consider a set E consisting of one point from the complement of each finite union, i.e., $x_n \notin G_1 \cup \cdots \cup G_n$. Since there are infinitely many finite unions and every point is in *some* set of the covering, the set E cannot be finite. (If $\{x_{i_1}, \ldots, x_{i_n}\}$ is any finite subset of E, there are sets $G_{j_1}, \ldots, G_{j_n}$ such that $x_{i_k} \in G_{j_k}$ for each k. Since E contains a point not in $G_{j_1} \cup \cdots \cup G_{j_n}$, it contains a point different from $x_1, \ldots, x_n$. Hence E is not finite.)

Now by hypothesis E must have a limit point z. The point z must belong to some set G_n; and since G_n is open, there is a number $\delta > 0$ such that $B_\delta(z) \subseteq G_n$. But then $B_\delta(z)$ cannot contain x_m if $m \geq n$, and so z cannot be a limit point of $\{x_m\}$. We have now reached a contradiction.

Exercise 2.27 Define a point p in a metric space X to be a *condensation point* of a set $E \subset X$ if every neighborhood of p contains uncountably many points of E.

Suppose $E \subset R^k$, E is uncountable, and let P be the set of all condensation points of E. Prove that P is perfect and that at most countably many points of E are not in P. In other words, show that $P^c \cap E$ is at most countable. *Hint:*

Let $\{V_n\}$ be a countable base of R^k, let W be the union of those V_n for which $E \cap V_n$ is at most countable, and show that $P = W^c$.

Solution. Following the hint, we see that $E \cap W$ is at most countable, being a countable union of at-most-countable sets. It remains to show that $P = W^c$, and that P is perfect.

If $x \in W^c$, and O is any neighborhood of x, then $x \in V_n \subseteq O$ for some n. Since $x \notin W$, $V_n \cap E$ is uncountable. Hence O contains uncountably many points of E, and so x is a condensation point of E. Thus $x \in P$, i.e., $W^c \subseteq P$.

Conversely if $x \in W$, then $x \in V_n$ for some V_n such that $V_n \cap E$ is countable. Hence x has a neighborhood (any neighborhood contained in V_n) containing at most a countable set of points of E, and so $x \notin P$, i.e., $W \subseteq P^c$. Hence $P = W^c$.

It is clear that P is closed (since its complement W is open), so that we need only show that $P \subseteq P'$. Hence suppose $x \in P$, and O is any neighborhood of x. (By definition of P this means $O \cap E$ is uncountable.) We need to show that there is a point $y \in P \cap (O \setminus \{x\})$. If this is not the case, i.e., if every point y in $O \setminus \{x\}$ is in P^c, then for each such point y there is a set V_n containing y such that $V_n \cap E$ is at most countable. That would mean that $y \in W$, i.e., that $O \setminus \{x\}$ is contained in W. It would follow that $O \cap E \subseteq \{x\} \cup (W \cap E)$, and so $O \cap E$ contains at most a countable set of points, contrary to the hypothesis that $x \in P$. Hence O contains a point of P different from x, and so $P \subseteq P'$. Thus P is perfect.

Remark: This result has now been proved to be true in any separable metric space, not just R^k.

Exercise 2.28 Prove that every closed set in a separable metric space is the union of a (possibly empty) perfect set and a set which is at most countable. (*Corollary:* Every countable closed set in R^k has isolated points.) *Hint:* Use Exercise 27.

Solution. If E is closed, it contains all its limit points, and hence certainly all its condensation points. Thus $E = P \cup (E \setminus P)$, where P is perfect (the set of all condensation points of E), and $E \setminus P$ is at most countable.

Since a perfect set in a separable metric space has the same cardinality as the real numbers, the set P must be empty if E is countable. The at-most-countable set $E \setminus P$ cannot be perfect, hence must have isolated points if it is nonempty.

Exercise 2.29 Prove that every open set in R^1 is the union of an at most countable collection of disjoint segments. *Hint:* Use Exercise 22.

Solution. Let O be open. For each pair of points $x \in O$, $y \in O$, we define an equivalence relation $x \sim y$ by saying $x \sim y$ if and only if $[\min(x, y), \max(x, y)] \subset O$. This is an equivalence relation, since $x \sim x$ ($[x, x] \subset O$ if $x \in O$); if $x \sim y$, then $y \sim x$ (since $\min(x,y) = \min(y,x)$ and $\max(x,y) = \max(y,x)$); and if $x \sim y$ and $y \sim z$, then $x \sim z$ ($[\min(x,z), \max(x,z)] \subseteq [\min(x,y), \max(x,y)] \cup [\min(y,z), \max(y,z)] \subseteq O$). In fact it is easy to prove that

$$\min(x,z) \geq \min\big(\min(x,y), \min(y,z)\big)$$

and

$$\max(x,z) \leq \max\big(\max(x,y), \max(y,z)\big).$$

It follows that O can be written as a disjoint union of pairwise disjoint equivalence classes. We claim that each equivalence class is an open interval.

To show this, for each $x \in O$, let $A = \{z : [z,x] \subseteq O\}$ and $B = \{z : [x,z] \subseteq O\}$, and let $a = \inf A$, $b = \sup B$. We claim that $(a,b) \subset O$. Indeed if $a < z < b$, there exists $c \in A$ with $c < z$ and $d \in B$ with $d > z$. Then $z \in [c,x] \cup [x,d] \subseteq O$. We now claim that (a,b) is the equivalence class containing x. It is clear that each element of (a,b) is equivalent to x by the way in which a and b were chosen. We need to show that if $z \notin (a,b)$, then z is not equivalent to x. Suppose that $z < a$. If z were equivalent to x, then $[z,x]$ would be contained in O, and so we would have $z \in A$. Hence a would not be a lower bound for A. Similarly if $z > b$ and $z \sim x$, then b could not be an upper bound for B.

We have now established that O is a union of pairwise disjoint open intervals. Such a union must be at most countable, since each open interval contains a rational number not in any other interval.

Exercise 2.30 Imitate the proof of Theorem 2.43 to obtain the following result:

> If $R^k = \cup_1^\infty F_n$, where each F_n is a closed subset of R^k, then at least one F_n has a nonempty interior.
>
> *Equivalent statement:* If G_n is a dense open subset of R^k, for $n = 1, 2, 3, \ldots$, then $\cap_1^\infty G_n$ is not empty (in fact, it is dense in R^k).

(This is a special case of Baire's theorem; see Exercise 22, Chap. 3, for the general case.)

Solution. The equivalence of the two statements is easily established. Suppose the first statement is true, and G_n is a dense open subset of R^k for $n = 1, 2, 3, \ldots$. Let $F_n = R^k \setminus G_n$. Then F_n is a closed subset of R^k having empty interior (if the interior of F_n were non-empty, G_n would not be dense). Hence by the first statement, the union of the set F_n cannot be all of R^k, and hence the intersection of their complements is not empty.

Conversely, if the second statement holds and F_n are closed subsets of R^k whose union is all of R^k, let G_n be the complement of F_n. Since the intersection of the G_n's is empty, at least one of them must fail to be dense in R^k, which means that its complement contains a non-empty open set.

We now prove the second statement, including the parenthetical remark. Let G_n be a sequence of dense open sets in R^k, and let O be any non-empty open set in R^k. Since O is an open set and G_1 is dense, it must intersect G_1 in a non-empty open set O_1. Let $x_1 \in O_1$, and choose $r_1 > 0$ such that the *closed* ball $\overline{B_{r_1}(x_1)}$ is contained in O_1. Then the *open* ball $B_{r_1}(x_1)$ is non-empty, and hence must intersect G_2 in a non-empty open set O_2. Let $x_2 \in O_2$, and choose $r_2 > 0$ such that the *closed* ball $\overline{B_{r_2}(x_2)}$ is contained in O_2. In this way we obtain a nested sequence of nonempty compact sets (closed balls) $\overline{B}_1 \supseteq \overline{B}_2 \supseteq \cdots \supseteq \overline{B}_n \supseteq \cdots$. If $x \in \cap\overline{B}_n$, then $x \in O_n$ for each n, and hence $x \in O \cap G_n$ for each n. Thus $\cap G_n$ intersects each non-empty open set O in at least one point, which says precisely that $\cap G_n$ is dense in R^k. Notice that the whole proof works exactly the same way if R^k is replaced by O, since $G_n \cap O$ is dense in O.

Remark: The stronger form of the second statement that we have proved shows that the first statement can also be strengthened. If $\{F_n\}$ is a sequence of closed sets whose union is all of R^k and O is any non-empty open set, then the interior of $F_n \cap O$ is non-empty for at least one n. (Simply apply the original statement with R^k replaced by O and F_k by $F_k \cap O$.)